

Hopf maps and Wigner's little groups

Ruben Mkrtchyan¹, Armen Nersessian² and Vahagn Yeghikyan³

¹*Yerevan Physics Institute, 2 Alikhanian Brothers St., Yerevan, 0036, Armenia*

²*Yerevan State University, 1 Alex Manoogian St., Yerevan, 0025, Armenia*

Abstract

We present the explicit formulae relating Hopf maps with Wigner's little groups. They, particularly, explain simple action of group on a fiber for the first and second Hopf fibrations, and present most simplified form for the third one. Corresponding invariant Lagrangians are presented, and their possible reductions are discussed.

1 Introduction

Hopf maps play distinguished role in modern theoretical physics: numerous constructions and models are related with them. Hopf maps are of the special importance in the supersymmetry [1, 2], monopoles [3], and more generally, in supergravity/string theories. Hopf maps are useful in the study of the problems of classical and quantum (supersymmetric) mechanics as well. They are useful for the construction of the mechanical systems (including supersymmetric ones) with monopoles by the reduction method, including supersymmetric mechanics. Particularly, the reduction related with the zero Hopf map yields the system with anyons, with the first Hopf map - to the systems with Dirac monopole, with the second Hopf map - with $SU(2)$ Yang monopoles (for the review see, e.g. [4] and refs therein). The detailed scheme of such a Lagrangian reduction related with the first/second Hopf maps, formulated in complex/quaternionic number's language has been developed in [5]. The reduction scheme based on the third Hopf map, and respectively, on octonionics, is not developed yet, to our knowledge.

The Hopf maps are closely related with supersymmetric theories in $(3+1)-$, $(5+1)-$ and $(9+1)$ dimensions. In this respect let us mention the paper written decade ago by Bandos, Lukierski and Sorokin [6], where massless superparticle model with tensorial central charges has been proposed and quantized. The authors of that paper noted the relation of their construction with Hopf maps, and also actually used the action of (subgroup of the) little group of momenta on corresponding spinor. Couple of years later one of the authors (R.M.) considered and explicitly constructed irreducible representations of tensorial Poincaré groups [7, 8] by Wigner's little group method [10]. Tensorial Poincaré groups are usual Poincaré, supplemented with additional tensorial central charges, corresponding to different supersymmetry algebras. It appears that exactly as quantization of particle in Minkowski space leads to irreps of Poincaré group, quantization of particle in tensorial spaces leads to the irreps of tensorial Poincaré groups. From the point of view of irreps of tensorial Poincaré, Hopf maps appear in the problem of decomposition of that irreps w.r.t. the usual Poincaré subgroup of tensorial Poincaré. Namely, when considering tensorial Poincaré in dimensions $(3+1)-$, $(5+1)-$ and $(9+1)-$, corresponding to the minimal $\mathcal{N} = 1$ supersymmetry algebras, taking a “preon” representation (i.e. those, corresponding to maximal number of surviving BPS supersymmetry), and decomposing that representations w.r.t. the usual Poincaré (i.e. considering particle content of preon), we get the Hopf fibrations of (unit norm) spinor spaces. In this way one conclude that the particle content of preon representation appears to be a tower of particles with all integer spins with multiplicity one. Note that we use here a notion of “preon” in a sense, introduced in [11], which denote a special configuration of tensorial central charges of supersymmetric theories, such that matrix of tensorial charges $Z_{\alpha\beta}$ has rank one, i.e. $Z_{\alpha\beta} \approx \lambda_\alpha \lambda_\beta$. As shown in [11], all other configuration of branes can be considered as some combinations of different number of preons, which explains the name of that object.

However, this tight connections between Hopf maps and the little groups of tensorial Poincaré had not been fully presented and studied up to now, as well as its differential-geometric formulation, while the latter is common language in the study of (super)particle mechanics and in the field theory. Moreover, as we noticed above, the (octonionic) reduction related with the third Hopf map, doesn't exist yet, and it is unclear, which sort of “monopole” should appear in the resulting system after such a reduction.

In the present paper we present the explicit differential geometrical description of the action of the $d = (3+1)$, $(5+1)$, $(9+1)$ -dimensional little groups, as well as establish their direct correspondence with Hopf maps. We also write down the metrics and one-forms, which are invariant under action of the little groups. Beside the above mentioned “practical” interest, our constructions are of the academic importance in the part, related with third Hopf map.

It is worth to mention hear the difference with the (closely related) twistorial approach to the same objects - Hopf maps, spinorial manifolds (twistors), etc (see a review on connection of twistors in different dimensions, division

algebras and Hopf maps in [12], and connected works [13]). As is clear from above, and will be exploited below, we are concentrated, first, on a connection between little groups and Hopf maps, actually obtaining the last object and its properties from the group theory approach, and, second, on a finite-dimensional mechanical systems and their reductions (as [4, 5]), based on corresponding manifolds.

2 General consideration

The description of Hopf maps can be given in terms of little groups (or stability groups) of Lorentz spinors. The generators of Lorentz group are given by $d \times d$ -dimensional antisymmetric matrices $\omega_{\mu\nu}$ and the transformation of vector V_μ is given by the following expression:

$$\delta V_\mu = \omega_{\mu\nu} V^\nu, \quad \mu, \nu = 1, \dots, d. \quad (1)$$

The transformation of corresponding spinor λ is given via $2^{d/2}$ -dimensional matrices γ^μ :

$$\delta \lambda = \omega_{\mu\nu} S^{\mu\nu} \lambda, \quad S^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu], \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad (2)$$

where $g^{\mu\nu}$ is the Minkowski's metrics. The operators $S^{\mu\nu}$ form $so(1, d-1)$ Lorentz algebra.

For some very special dimensionalities d the space of spinors (without zero) of d -dimensional Lorentz group $SO(1, d-1)$ can be represented as factor space of this group by stability subgroup of one given spinor:

$$\{\lambda\} = SO(1, d-1)/\mathcal{G}, \quad g\lambda_0 = \lambda_0, \quad g \in \mathcal{G}, \lambda_0 \in \{\lambda\} \quad (3)$$

Calculations of these subgroups and dimensionalities can be found in [7], here we are interested only in the results of that paper for fixed dimensions where Hopf maps exist, so from now on $d = 3+1, 5+1, 9+1$. The corresponding stability groups are:

$$\mathcal{G}_{1,3} = T^2 \quad (4)$$

$$\mathcal{G}_{1,5} = SO(3) \ltimes T^4 \quad (5)$$

$$\mathcal{G}_{1,9} = SO(7) \ltimes T^8, \quad (6)$$

where T^k is k -dimensional translational subgroup of corresponding Lorentz group. Here \ltimes denotes semidirect product.

Now for every spinor λ one can define d quantities p^μ (components of vector) which will transform as in 1,

$$p^\mu = \bar{\lambda} \gamma^\mu \lambda, \quad \bar{\lambda} = \lambda^* A, \quad (7)$$

where $*$ means complex conjugation and $A = \gamma^0$. From this expression it follows, that the vector p^μ is light-like one,

$$p_\mu p^\mu = 0. \quad (8)$$

On the other hand, one can consider all Lorentz transformations which leave invariant the quantities p^μ , i.e. the matrixes $\omega_{\mu\nu}$, which satisfy an equation

$$\omega_{\mu\nu} p^\nu = 0. \quad (9)$$

Resolving this equation, we get

$$\begin{cases} \omega_{0i} p^i = 0 \\ \omega_{i0} p^0 + \omega_{ij} p^j = 0 \end{cases} \Rightarrow \omega_{0i} = -\omega_{i0} = \frac{\omega_{ij} p^j}{p_0} \quad (10)$$

It is obvious that above considered little groups of spinors are subgroups of these transformations. Indeed, any transformation, which does not change spinors λ will also leave invariant vector-quantities p^μ . For the considered dimensions this transformations are:

$$d = 3+1 : \quad SO(2) \ltimes T^2 \quad (11)$$

$$d = 5+1 \quad SO(4) \ltimes T^4 \quad (12)$$

$$d = 9+1 \quad SO(8) \ltimes T^8 \quad (13)$$

Which transformations change the spinor λ but leave invariant the quantities p_μ ? It is clear, that they form the bundle of fibration of spinor space over the space of quantities p^μ . For the considered cases we have

$$\begin{aligned} SO(2) \otimes T^2/T^2 &= SO(2) \equiv S^1 \\ SO(4) \otimes T^4/SO(3) \otimes T^4 &= SO(3) \equiv S^3 \\ SO(8) \otimes T^8/SO(7) \otimes T^8 &= S^7 \end{aligned} \quad (14)$$

One can prove (at least, for the special representation of γ^μ matrices) that fixation $2^{d/2} - 1$ -dimensional sphere in the space of spinors leads to fixation of $(d - 2)$ -dimensional sphere in the space of p^μ :

$$(\lambda^*)^T \lambda = p_i p_i = 1, \quad i = 1, \dots, d - 1. \quad (15)$$

Let us notice, that this identity is not Lorentz-invariant one, and depends on the realization of matrices γ^μ . On the other hand, the existence of this identity in one realization leads to the existence of similar identity in any other one, which can be obtained by unitary transformation. Because of non-invariant form of (15), the points of spheres in different realizations are not coincide.

So, taking in mind the bundles (14) we get factorizations of spheres over spheres,

$$S^3/S^1 = S^2, \quad S^7/S^3 = S^4, \quad S^{15}/S^7 = S^8, \quad (16)$$

i.e. the first, second and third Hopf maps. We presented quite simple procedure for the describing Hopf maps. However, the considered transformations are too general and not so easy to use. Indeed, the matrices $\omega_{\mu\nu}$ obey the condition (9), which decreases the number of $d(d - 1)/2$ parameters by $d - 1$ only. This matrix depends on the components p_μ and, therefore, on spinor λ . Hence, transformations (1) are non-linear ones. On the other hand, any linear transformation $\omega \mapsto f(\omega, \lambda)$, where f is an arbitrary function, does not change the condition (9).

Whether it is possible to choose the form of $\omega_{\mu\nu}$ simplifying the above transformation (2) to the linear one? For answering on this question, in the next sections we shall consider the transformation (2) separately for the $d = 3 + 1, 5 + 1, 9 + 1$ dimensions. However, before going to do that, we will consider, in the next section, the relation of the Hopf map with division algebras.

3 Hopf maps

The Hopf maps $S^{2n-1}/S^{n-1} = S^n$, $n = 1, 2, 4, 8$ are closely related with the division algebras. Precisely, zero Hopf map $n = 1$ reflects the existence of division algebra of real numbers, first Hopf map $n = 2$ - complex numbers, second Hopf map $n = 4$ - quaternions, third Hopf map $n = 8$ - octonions.

Let us describe the Hopf maps in explicit terms, demonstrating their relation with the division algebras. For this purpose, we consider the functions

$$\mathbf{p} = 2\bar{\mathbf{u}}_1 \mathbf{u}_2, \quad p_{n+1} = \bar{\mathbf{u}}_1 \mathbf{u}_1 - \bar{\mathbf{u}}_2 \mathbf{u}_2, \quad (17)$$

where $\mathbf{u}_1, \mathbf{u}_2$ are real numbers for the $n = 1$ case (zero Hopf map), complex numbers for the $n = 2$ case (first Hopf map), quaternionic numbers for the $n = 4$ case (second Hopf map) and octonionic numbers for the $n = 8$ case (third Hopf map). One can consider them as coordinates of the $2n$ -dimensional space \mathbb{R}^{2n} ($n = 1$ for real $\mathbf{u}_{1,2}$; $n = 2$ for complex $\mathbf{u}_{1,2}$; $n = 4$ for quaternionic $\mathbf{u}_{1,2}$; $n = 8$ for octonionic $\mathbf{u}_{1,2}$). In all cases p_{n+1} is a real function, while \mathbf{p} is, respectively, real function ($n = 1$), complex function ($n = 2$), a quaternionic one ($n = 4$), and octonionic one ($n = 8$),

$$\mathbf{p} \equiv p_n + \sum_{k=1, \dots, n-1} \mathbf{e}_k p_k, \quad \mathbf{e}_i \mathbf{e}_j = -\delta_{ij} + C_{ijk} \mathbf{e}_k \quad (18)$$

where the structure constants C_{ijk} are totally antisymmetric by indices (ijk) , so that $\mathbf{e}_k \equiv 0$ for $n = 1$; $\mathbf{e}_k \equiv \mathbf{i}$, $\mathbf{i}^2 = -1$, $c_{ijk} = 0$ for $n = 2$; $\mathbf{e}_k \equiv (\mathbf{i}, \mathbf{j}, \mathbf{k})$, $C_{ijk} = \varepsilon_{ijk}$ for $n = 4$. For $n = 8$ the structure constants c_{ijk} are defined by the relations

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1, \quad (19)$$

while all other non-vanishing components are determined by the total antisymmetry. Hence, (p_{n+1}, \mathbf{p}) parameterize the $(n + 1)$ -dimensional space \mathbb{R}^{n+1} .

One can check immediately, that the following equation holds:

$$p_0^2 \equiv \bar{\mathbf{p}} \mathbf{p} + p_{n+1}^2 = (\bar{\mathbf{u}}_1 \mathbf{u}_1 + \bar{\mathbf{u}}_2 \mathbf{u}_2)^2. \quad (20)$$

Thus, defining the $(2n - 1)$ -dimensional sphere with radius $\sqrt{p_0}$, $\bar{\mathbf{u}}_\alpha \mathbf{u}_\alpha = p_0$ in \mathbb{R}^{2p} , we will get the p -dimensional sphere with radius p_0 in \mathbb{R}^{p+1} .

The expressions (17) can be easily inverted by the use of equality (20)

$$\mathbf{u}_\alpha = \mathbf{g} r_\alpha, \quad \text{where} \quad r_1 = \sqrt{\frac{p_0 + p_{p+1}}{2}}, \quad r_2 \equiv r_+ = \frac{\mathbf{p}}{\sqrt{2(p_0 + p_{p+1})}}, \quad \bar{\mathbf{g}} \mathbf{g} = 1. \quad (21)$$

It follows from the last equation in (21) that \mathbf{g} parameterizes the $(p - 1)$ -dimensional sphere of unit radius.

Using above equations, it is easy to describe the first three Hopf maps. Indeed, for $n = 1, 2, 4$ the functions \mathbf{p}, p_{n+1} remain invariant under the transformations

$$\mathbf{u}_\alpha \rightarrow \mathbf{G} \mathbf{u}_\alpha, \quad \text{where} \quad \bar{\mathbf{G}} \mathbf{G} = 1 \quad (22)$$

Therefore, \mathbf{G} parameterizes the spheres S^{n-1} of unit radius. Taking into account the isomorphism between these spheres and the groups for $n = 1, 2, 4$ ($S^0 = Z_2$, $S^1 = U(1)$, $S^3 = SU(2)$), we get that (17) is invariant under G -group transformations for $n = 1, 2, 4$ (where $G = Z_2$ for $n = 1$, $G = U(1)$ for $n = 2$, and $G = SU(2)$ for $n = 4$).

It is easy to construct the one-forms and metrics, which are invariant under “global” G -action (22), i.e. for $\mathbf{G} = \text{const}$:

$$\omega_1^0 = \text{Re } f_\mu(\mathbf{p})(\bar{\mathbf{u}} \sigma^\mu d\mathbf{u}) + \text{Re } \lambda_{\mu\nu}(p)(\bar{\mathbf{u}} \sigma^\mu \bar{\mathbf{u}})(\mathbf{u} \sigma^\nu d\mathbf{u}), \quad (23)$$

$$(ds)^2 = f_\mu(p)(d\bar{\mathbf{u}} \sigma^\mu d\mathbf{u}) + f_{\mu\nu}(p)(\bar{\mathbf{u}} \sigma^\mu d\mathbf{u})(d\bar{\mathbf{u}} \sigma^\nu \mathbf{u}) + \text{Re } \lambda_{\mu\nu}(p)(\bar{\mathbf{u}} \sigma^\mu \bar{\mathbf{u}})(d\mathbf{u} \sigma^\nu d\mathbf{u}), \quad (24)$$

where $f_{\mu\nu}(p) = \bar{f}_{\nu\mu}(p)$, and $f_\mu(p), \lambda_{\mu\nu}$ are arbitrary complex functions.

The Lagrangians constructed by the use of the above one-form and metric, will also be G -invariant one. Using Hopf maps, they can be easily reduced to the low dimensional systems both in the Hamiltonian and Lagrangian framework. It is worth to mention here that spinorial origin of quantities in Lagrangians doesn't lead to any spin-statistics problems both in this non-relativistic case (where one don't expect them due to non-relativistic framework), as well as in relativistic case, see e.g. [6], where bosonic spinors essentially play the role of source of constraints.

For $n = 1$ case ($G = Z_2$, zero Hopf map) we simply reformulate two-dimensional system in terms Z_2 -invariant coordinates, i.e. perform (locally) equivalence transformation. Nevertheless, the reduction by Z_2 group has the consequences if we consider global properties of the system. It becomes visible on the quantum-mechanical level, and yields the system interacting with the anyons (see, e.g., [14]).

For the $n = 2$ case ($G = U(1)$, first Hopf map) the reduction results the initial four-dimensional system in the three-dimensional one interacting with Dirac monopole. For the particular examples of such a (Hamiltonian) reductions, we refer to [15, 16] and references therein.

For the $n = 4$ case ($G = SU(2)$, second Hopf map) the reduction is more complicated, due to non-Abelian nature of $SU(2)$ group. It results the initial eight-dimensional system in the five-dimensional one, interacting with $SU(2)$ Yang monopole. respectively, the reduced system has twelve-dimensional phase space $T^*R^5 \times S^2$. For the particular example of such a Hamiltonian reduction we refer to [17, 18] and references therein. Let us also mention the recent paper [5], where the detail procedure of the Lagrangian reduction related with first and second Hopf map has been developed on the simplest example of the $2n$ -dimensional Lagrangian $\mathcal{L}_0 = g(\bar{\mathbf{u}} \cdot \mathbf{u}) \dot{\bar{\mathbf{u}}}_\alpha \dot{\mathbf{u}}_\alpha$. Its extension to the Lagrangians associated with metric (24) is straightforward.

For the octonionic case $n = 8$ situation is more complicated. Because of losing associativity the standard transformation of \mathbf{u}_α that leaves invariant coordinates \mathbf{p}, p_9 will not be just (22). Its modification can be easily obtained using (21):

$$\mathbf{u}_\alpha \mapsto (\mathbf{G} \mathbf{g})(\bar{\mathbf{g}} \mathbf{u}_\alpha) = \frac{(\mathbf{G} \mathbf{u}_1)(\bar{\mathbf{u}}_1 \mathbf{u}_\alpha)}{\bar{\mathbf{u}}_1 \mathbf{u}_1}. \quad (25)$$

Also, for the $n = 8$ case the fiber S^7 is not isomorphic with any group. So, it is unclear, how to construct the octonionic analog of the one-form (23) and metric (24), which will be invariant under transformation (25). So, one can expect further troubles in the extensions of the constructions, related with the lower Hopf maps to the third one. As a consequence, the extension of the above-described reduction procedures to the octonionic case is an open problem yet.

4 $d = 3 + 1$ case and first Hopf map

Let us consider the action of the Wigner's little group of momenta on the corresponding spinors, in the $d = 3 + 1$ dimension. For this purpose it is convenient to choose a specific realization of γ -matrices,

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ (\sigma^i)^T & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix}, \quad A = \gamma^0, \quad i = 1, 2, 3. \quad (26)$$

Upon this realization, for the Majorana spinor λ the expressions for momenta (7) read

$$p^\mu = \lambda^* \gamma^0 \gamma^\mu \lambda, \quad (27)$$

where

$$\lambda = \begin{pmatrix} z_1 \\ z_2 \\ \bar{z}_2 \\ -\bar{z}_1 \end{pmatrix} \quad (28)$$

Hence, one can replace the expressions (27) with the following ones:

$$p^0 = Z\bar{Z}, \quad p^i = Z\sigma^i\bar{Z}, \quad \lambda = (Z, \sigma^1\bar{Z}) \quad Z = (z_1, z_2), \quad \bar{Z} = Z^* \quad (29)$$

In that case the solutions of the equations (10) can be represented as follows :

$$\omega_{0i} = \epsilon_{ijk} a^j p^k, \quad \omega_{ij} = \epsilon_{ijk} a^k p^0, \quad (30)$$

while corresponding transformation of spinor λ (2), will take a form

$$\delta\lambda = \omega_{\mu\nu} \gamma^{\mu\nu} \lambda = 2\omega_{0i} \gamma^{0i} + \omega_{ij} \gamma^{ij}. \quad (31)$$

Here

$$\gamma^{0i} = \frac{[\gamma^0, \gamma^i]}{2} = \begin{pmatrix} -\sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \quad \gamma^{ij} = \frac{[\gamma^i, \gamma^j]}{2} = \begin{pmatrix} -i\epsilon^{ijk}\sigma_k & 0 \\ 0 & -i\epsilon^{ijk}\sigma_k \end{pmatrix} = i\epsilon^{ijk}\gamma_{0i}. \quad (32)$$

Equivalently,

$$\delta Z = 2\omega_{0i}\sigma^i Z + \omega_{ij}\epsilon^{ijk}\sigma_k Z \quad (33)$$

Substituting the solutions from (30) in the second summand, and transforming the first one by the use of Fierz identity

$$i\epsilon^{ijk}(u\sigma_j\bar{v})(r\sigma_k\bar{s}) = (u\bar{s})(r\sigma_k\bar{v}) - (r\bar{v})(u\sigma_i\bar{s}), \quad (34)$$

we get

$$\delta Z = i a^i (Z\sigma_i\bar{Z})Z \quad (35)$$

Since a_i are arbitrary functions depending on Z, \bar{Z} , we can represent (35), upon their proper redefinitions, as follows

$$\delta Z^\alpha = i a(Z, \bar{Z}) Z^\alpha, \quad (36)$$

with $a(Z, \bar{Z})$ being arbitrary function. The respective finite transformation reads $Z \rightarrow e^{i\varphi(Z, \bar{Z})} Z$. This is a phase shift, so we get that the little group of momenta on corresponding four dimensional Lorenz spinor is $U(1) = S^1$, parametrising the sphere of unit radius. The action of the little group has a natural geometric interpretation in terms of first Hopf map. Indeed, the relation $\bar{Z}Z = \sqrt{p_i p_i}$, upon fixing of value of $\bar{Z}Z = p_0 = \text{const}$, yields the fibration corresponding to the first Hopf map $S^3/S^1 = S^2$ (see previous Section). So, the spinors can be formulated in the coordinates associated with Hopf fibration (21), viz

$$Z^\alpha = \left(e^{i\gamma} \sqrt{\frac{p_0 + p_3}{2}}, \quad e^{i\gamma} \frac{(p_1 + ip_2)}{\sqrt{2(p_0 + p_3)}} \right), \quad (37)$$

where the angle $\gamma \in [0, 2\pi)$ parameterizes the fiber S^1 . Thus, the transformation (36) acts on the fiber coordinate γ only: $\gamma \rightarrow \gamma + a(\gamma, p)$.

There is variety of ways to construct the Lagrangians which are invariant under this, "local" $U(1)$ transformations.

For example, one can construct the one-forms, which are invariant under (36):

$$\omega^1 = \imath f_j(\bar{Z}\sigma^j dZ) + \lambda_{ij}(p)(\bar{Z}\sigma^i \bar{Z})(Z\sigma^j dZ) + \text{c.c.}, \quad (38)$$

where $f_i(p)$ and $\lambda_{ij}(p)$ are real symmetric tensors. The respective first-order Lagrangian will be invariant under above transformation as well.

One can also construct the metric which is invariant with respect to (36) transformation:

$$ds^2 = ds_0^2 + ds_1^2, \quad (39)$$

where

$$ds_0^2 = \imath b_{ij}(p) [p^i (d\bar{Z}\sigma^j dZ) - (d\bar{Z}\sigma^i Z)(\bar{Z}\sigma^j dZ)], \quad b_{ij}(p) = b_{ji}(p) \quad (40)$$

$$ds_1^2 = f_{lkij} [\text{Re} (\bar{Z}\sigma^l \bar{Z})(\bar{Z}\sigma^k \bar{Z})(Z\sigma^i dZ)(Z\sigma^j dZ) - 2(Z\sigma^l Z)(\bar{Z}\sigma^k \bar{Z})(Z\sigma^i dZ)(\bar{Z}\sigma^j \bar{Z})], \quad (41)$$

where $f_{lkij}(p) = f_{klij}(p)$, $f_{lkij}(p) = f_{lkji}(p)$, and is p^i expressed via \bar{Z} , Z in accordance with (27). It is clear, that by the use of this metric we can construct reparametrization-invariant Lagrangian $\mathcal{L} = |ds/dt|$ and sigma-model-type Lagrangian as well, $\mathcal{L} = ds^2/dt^2$. One can also construct the higher-derivative Lagrangians, which are invariant under (36): these Lagrangians will be the functions of extrinsic curvatures associated with the above constructed metrics. Surely, such generic Lagrangians are not invariant under action of the whole Lorentz group, but under its little group only.

It is easy to see, that both particular metrics, (40) and (41) as well as their sum (39), are degenerated one: the vector field $\mathbf{V} = Z^\alpha \partial_\alpha + \text{c.c.}$ defines their null vector. Hence, the respective Lagrangians will be degenerated: they will have constraint defined by this null vector. In fact, these Lagrangians could be reformulated as a non-singular ones depending on momenta \mathbf{p} only.

On the other hand, instead of generic local transformation (36) one can consider particular one, when $a \equiv p_i a_i(Z, \bar{Z})$ is a constant number. For this purpose one can choose, e.g

$$a_i(Z, \bar{Z}) = \frac{c_i}{p_i} \text{ (with no summation over } i), \quad c_i = \text{const.} \quad (42)$$

In that case the action of the little group on the spinor will be multiplication on the constant phase factor. The reduction of the four-dimensional systems under such $U(1)$ group action is widely known in classical and quantum mechanics, and yields the systems interacting with Dirac monopole (see, e.g. [4, 5]).

5 $d = 5 + 1$ case and second Hopf map

In this Section we consider the action of little group of momenta on the corresponding spinors in $d = 5 + 1$ dimensional space. Since in $d = 5 + 1$ case there is no Majorana spinor, we should use a Weyl one,

$$\lambda = (z_1, z_2, z_3, z_4, 0, 0, 0, 0). \quad (43)$$

Similar to the previous case, we define

$$p^\mu = \bar{\lambda} \Gamma^\mu \lambda, \quad \text{where } \bar{\lambda} = \lambda^* \Gamma^0, \quad \Gamma^0 = \begin{pmatrix} 0 & \imath \mathbf{1}_4 \\ -\imath \mathbf{1}_4 & 0 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ -\gamma^i & 0 \end{pmatrix}, \quad i = 1, \dots, 5. \quad (44)$$

Then we introduce

$$Z = (z_1, z_2, z_3, z_4), \quad \bar{Z} = Z^*, \quad (45)$$

and rewrite the expression for momentum p_μ (44) in the following form:

$$p^0 = Z \bar{Z}, \quad p^i = -\bar{Z} \gamma^i Z \quad (46)$$

Now, for the transformation for spinor λ we have:

$$\delta \lambda = \omega_{\mu\nu} \Gamma^{\mu\nu} \lambda = 2 \frac{\omega_{ij} p^j}{p^0} \Gamma^{0i} \lambda + \omega_{ij} \Gamma^{ij} \lambda, \quad (47)$$

where

$$\Gamma^{0i} = \frac{[\Gamma^0, \Gamma^i]}{2} = \begin{pmatrix} -\gamma^i & 0 \\ 0 & -\gamma^i \end{pmatrix}, \quad \Gamma^{ij} = \frac{[\Gamma^i, \Gamma^j]}{2} = \begin{pmatrix} -\gamma^{ij} & 0 \\ 0 & -\gamma^{ij} \end{pmatrix}, \quad (48)$$

where γ^i are Euclidean gamma-matrices:

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \quad (49)$$

It is easy to see, that one can replace the transformation formula for λ (47) with the following one

$$\delta Z = -2 \frac{\omega_{ij}(\bar{Z}\gamma^j Z)}{p_0} \gamma^i Z + \omega_{ij} \gamma^{ij} Z \quad (50)$$

For further manipulations with this expression we will use the Fierz identity

$$Z_\alpha \bar{Z}_\beta = \frac{1}{4}(Z\bar{Z})\delta_{\alpha\beta} + \frac{1}{4}(\bar{Z}\gamma^l Z)\gamma_{\alpha\beta}^l - \frac{1}{8}(\bar{Z}\gamma^{lm} Z)\gamma_{\alpha\beta}^{lm}. \quad (51)$$

Applying this identity to (50), we get, after proper simplification,

$$\omega_{ij}(\bar{Z}\gamma^j Z)\gamma^i Z = -(\bar{Z}\gamma^{ij} Z)\omega_{ij} Z - (\bar{Z}\gamma^{lj} Z)\omega_{ij}\gamma^{il} Z + (\bar{Z}Z)\omega_{ij}\gamma^{ij} Z \quad (52)$$

Applying the same identity to the second term of this expression, we get the final expression for the transformation law of the spinor Z

$$\delta Z = -(\bar{Z}\gamma^{ij} Z)\omega_{ij} Z + (ZC\gamma^{ij} Z)\omega_{ij} C\bar{Z}, \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad (53)$$

Here we used identities

$$(\bar{Z}\gamma^l Z)\omega_{ij}\gamma^{ij}\gamma^l Z = (\bar{Z}Z)\omega_{ij}\gamma^{ij} Z, \quad \frac{1}{4}(\bar{Z}\gamma^{lm} Z)\omega_{ij}\gamma^{ij}\gamma^{lm} Z = -(\bar{Z}Z)\omega_{ij}\gamma^{ij} Z. \quad (54)$$

To make the origin of transformation (53) more transparent, let us reformulate it in terms of quaternions. Taking into account the definition of spinor Z and the block-diagonal form of matrix C we introduce the notation

$$\mathbf{u}_1 = z_1 + z_2 \mathbf{j}, \quad \mathbf{u}_2 = z_3 + z_4 \mathbf{j}, \quad \boldsymbol{\tau} = (\bar{Z}\gamma^{ij} Z)\omega_{ij} + (ZC\gamma^{ij} Z)\omega_{ij} \mathbf{j}, \quad (55)$$

In these terms the transformation (53) reads

$$\delta \mathbf{u}_\alpha = \boldsymbol{\tau} \mathbf{u}_\alpha. \quad (56)$$

Since $\boldsymbol{\tau}$ is pure imaginary quaternionic function, we conclude, that finite transformation corresponding to (56) is of the form

$$\mathbf{u}_\alpha \rightarrow \mathbf{G}(\mathbf{u}, \bar{\mathbf{u}}) \mathbf{u}_\alpha, \quad \mathbf{G}\bar{\mathbf{G}} = 1, \quad (57)$$

i.e. we arrived $SO(3) = S^3$ transformation given by (22).

Similar to the $d = 3 + 1$ case, with an appropriate redefinition of the parameters ω_{ij} , the transformation (56) (and (57)) can be “flattened”.

The metrics, which are invariant under such a “local” $SU(2)$ transformation will be defined by the same formulae, as in the $d = 3 + 1$ case, (40),(41), where the complex spinor Z^α is replaced by the quaternionic one \mathbf{u}^α .

The reductions of the Lagrangians constructed by the use of this $SU(2)$ -invariant metric could be done following the receipt presented in [5] for the reduction by the action of the “flat” $SU(2)$ transformation. In contrast with $d = 3 + 1$ case, the reduced system will be depended not only on momenta p_i , but from the two spin variables as well.

6 $d = 9 + 1$ and third Hopf map

Now, let us consider the action of the little group of momentum for the last, $d = 9 + 1$ dimensional case, corresponding to the third Hopf map.

For the $SO(1, 9)$ algebra the minimal spinor is a Majorana-Weyl one. Choosing the chiral representation of Γ -matrices

$$\Gamma^0 = \begin{pmatrix} 0 & i\mathbf{1}_4 \\ -i\mathbf{1}_4 & 0 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix} \quad (58)$$

one can write the 32-spinor λ in the following form

$$\lambda = (Z_1, C_4 \bar{Z}_1, Z_2, C_4 \bar{Z}_2, 0, \dots, 0), \quad \text{where } Z_1 = (z_1, z_2, z_3, z_4), \quad Z_2 = (z_5, z_6, z_7, z_8) \quad (59)$$

are general 4-dimensional Dirac spinors and C_4 is (4×4) C-matrix for dimensionality $4 + 1$.

Similar to the previous sections, we define the quantities p_μ by the expression (44). Denoting the first sixteen elements of spinor λ by Z , we will get, for the quantities p_μ , the realization (46).

Then, performing manipulations, which are completely similar to those in the Fourth and Fifth Sections, we shall get the following expression for the transformations, which leave invariant the momenta p_μ :

$$\delta Z = -2 \frac{\omega_{ij}(\bar{Z} \gamma^j Z)}{p_0} \gamma^i Z + \omega_{ij} \gamma^{ij} Z \quad (60)$$

This expression can be transformed using Fierz reordering formula. Namely, for any (16×16) matrix A we have following decomposition

$$A = \frac{1}{16} \text{Tr}(A) \mathbf{1}_{16} + \frac{1}{16} \text{Tr}(A \gamma^i) \gamma^i - \frac{1}{32} \text{Tr}(A \gamma^{ij}) \gamma^{ij} - \frac{1}{96} \text{Tr}(A \gamma^{ijk}) \gamma^{ijk} + \frac{1}{384} \text{Tr}(A \gamma^{ijkl}) \gamma^{ijkl}, \quad (61)$$

where

$$\gamma^{i_1 i_2 \dots i_k} = \frac{1}{k!} \gamma^{[i_1} \gamma^{i_2} \dots \gamma^{i_k]} \quad (62)$$

Applying twice this identity to (60), we get the simple final expression

$$\delta Z = -\frac{1}{6} \omega_{ij} (Z C \gamma^{ijlm} Z) \gamma^{lm} Z \quad (63)$$

Taking into account general considerations of the Second Section, we conclude, that (63) defines the S^7 action in the fiber of third Hopf fibration $S^{15}/S^7 = S^8$. Thus, being reformulated in octonionic terms, it could be represented in the form

$$\mathbf{u}_\alpha \mapsto (\tau \mathbf{g})(\bar{\mathbf{g}} \mathbf{u}_\alpha) = (\tau \mathbf{u}_1)(\bar{\mathbf{u}}_1 \mathbf{u}_\alpha), \quad \text{Re } \tau = 0, \quad (64)$$

which is an infinitesimal form of (25). In contrast with (35) and (53), transformation (63) does not possess Lie-algebraic structure, but form some quadratic algebra. This is in accordance with the fact, that S^7 is not a group manifold. It is clear from the expression (64), that this transformation can not be flattened in analogy with (42). Consequently, it is not so clear for us, how to construct the octonionic (S^7 -invariant) analogs of the one-form (23) and metric (24) (which are invariant under “flat” S^1/S^3 transformations). The extension of the reduction procedure developed in [5], to the given octonionic case is also not so straightforward. In any case, consideration of these problems should definitely be easier to consider in the complex terms (63), than in octonionic ones. We suppose to consider them elsewhere.

7 Conclusion and Outlook

We show how Hopf maps are connected to the little groups of “preon” orbits in relevant dimensions $1 + 3$, $1 + 5$ and $1 + 9$. Simple realization of Hopf maps for first two cases, known earlier [4], is recovered from the present approach. The unsolved challenge remains construction of (necessarily non-quadratic) Lagrangian, suited for the reduction, realizing third Hopf map, which will bring to the still unknown mechanical system, generalizing those with monopoles for lower dimensions. For that case we get a simple expressions for the action of little group on a S^7 fiber, which presumably will help in construction of Lagrangian, necessarily non-quadratic, realizing the third Hopf map. Another direction of study may be to generalize Poincaré, i.e. to substitute Lorentz group, as well as

tensorial central charges, with some more general objects, containing those as a subgroups, with purpose to obtain generalization of little groups and corresponding Hopf maps. Very deep such a generalization is known already few years, that is a Kac-Moody approach to superstrings/supergravity theories (see [19], [20]). For our purposes we need an analog of our spinor λ in that approach. Its explicit description appears to be one of the main problems of that theories, such a “spinor representation” would be a supersymmetry charge, important for construction of full supersymmetric theory. First few terms of decomposition of that charge w.r.t. the Lorentz group are known, only.

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